

C24 Dynamical Systems

Lecture 8: Chaos

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Lecture 8 overview

- Now we shift attention away from differential equations and consider discrete time systems or maps:

$$\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k)$$

- We will investigate the behaviour of these maps graphically using cobweb plots
- Unlike 1-D differential systems, the discrete 1-D map can exhibit the phenomenon of chaos; a few mathematical techniques will help analyze such systems
- Finally we return to differential systems, examining the chaotic Lorenz equations and identifying a strange attractor

Analysis of fixed points of maps

- Given a discrete map $\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k)$, a fixed point \mathbf{x}^* satisfies

$$\mathbf{x}^* = \mathbf{F}(\mathbf{x}^*)$$

- The map's behaviour near the equilibrium point \mathbf{x}^* can be characterised by linearising \mathbf{F} :

$$\begin{aligned}\mathbf{x}^* + \mathbf{w}_{k+1} &= \mathbf{F}(\mathbf{x}^*) + D\mathbf{F}(\mathbf{x}^*)\mathbf{w}_k + \dots \\ \implies \mathbf{w}_{k+1} &= D\mathbf{F}(\mathbf{x}^*)\mathbf{w}_k + \dots\end{aligned}$$

- Stability can be assessed by analysing properties of the Jacobian $D\mathbf{F}$ at the equilibrium point
- Note that for the 1-D case the Jacobian of the map is simply

$$DF(x^*) = \left. \frac{dF}{dx} \right|_{x^*}$$

Stability of fixed points

- For a linear map the stability of the fixed point at the origin can be understood by diagonalisation (q2 on example sheet 1)

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k \quad \Longrightarrow \quad \mathbf{x}_n = \mathbf{A}^n \mathbf{x}_0 \quad \Longrightarrow \quad \mathbf{x}_n = \mathbf{V}\mathbf{\Lambda}^n \mathbf{V}^{-1} \mathbf{x}_0$$

- The behaviour as $n \rightarrow \infty$ determines the stability of the map, e.g. in 2-D

$$\lim_{n \rightarrow \infty} \mathbf{\Lambda}^n = \lim_{n \rightarrow \infty} \begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix} \quad \begin{array}{l} |\lambda|, |\mu| > 1 \implies \text{unstable node} \\ |\lambda|, |\mu| < 1 \implies \text{stable node} \\ |\lambda| > 1, |\mu| < 1 \implies \text{saddle} \end{array}$$

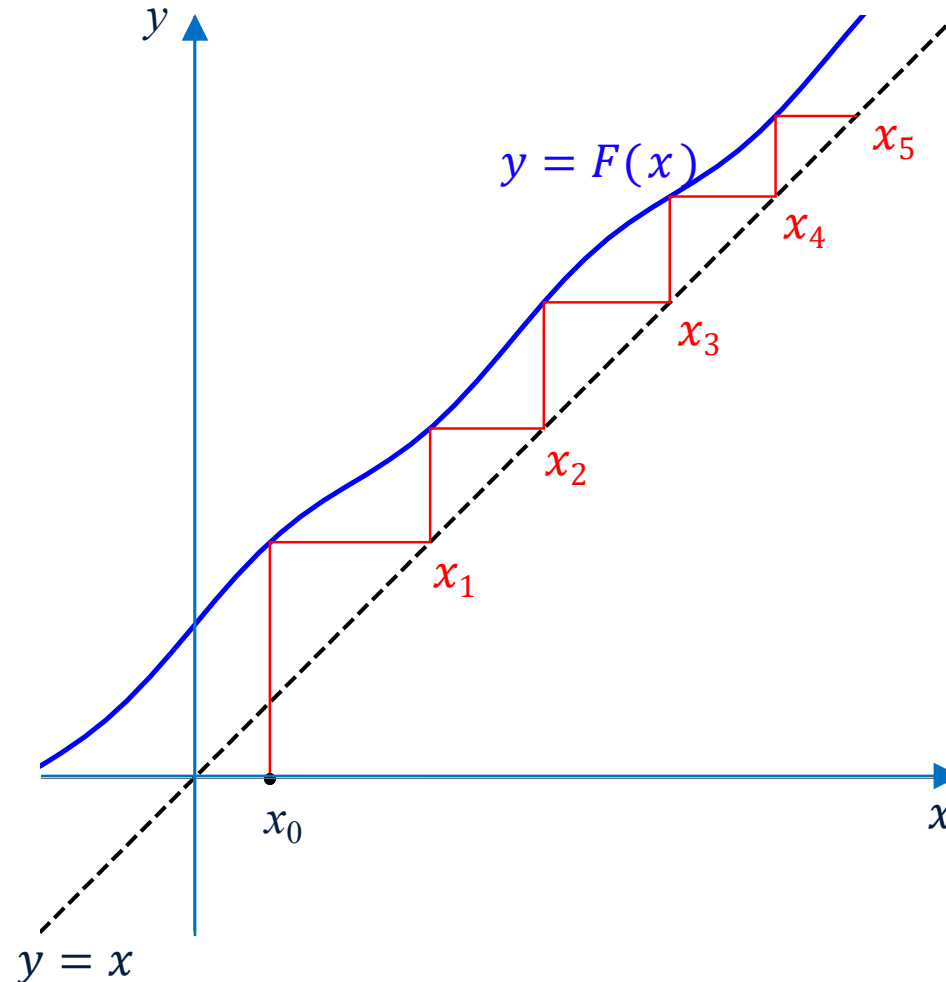
- Eigenvalues inside the unit circle are stable
- Hence for the 1-D case x^* is a stable fixed point if $|DF(x^*)| < 1$

Cobweb diagrams

- Cobweb diagrams represent graphically the iterates of a map

- Given map $x_{k+1} = F(x_k)$, use the following procedure:

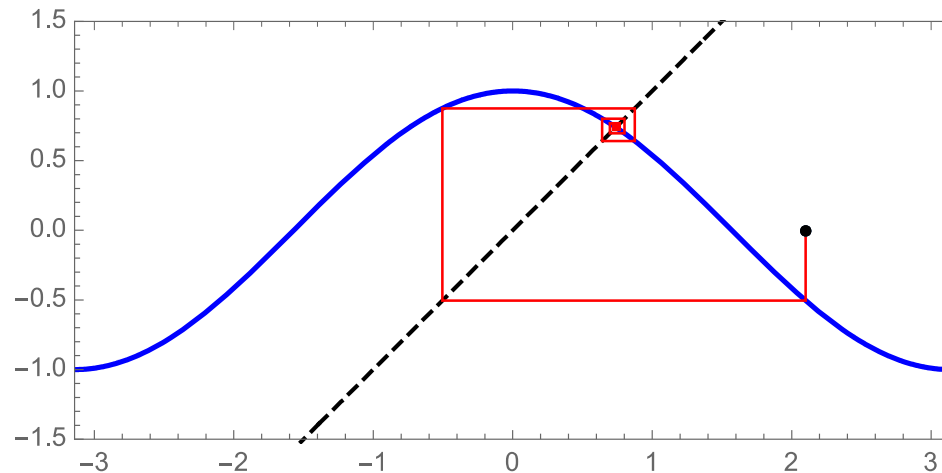
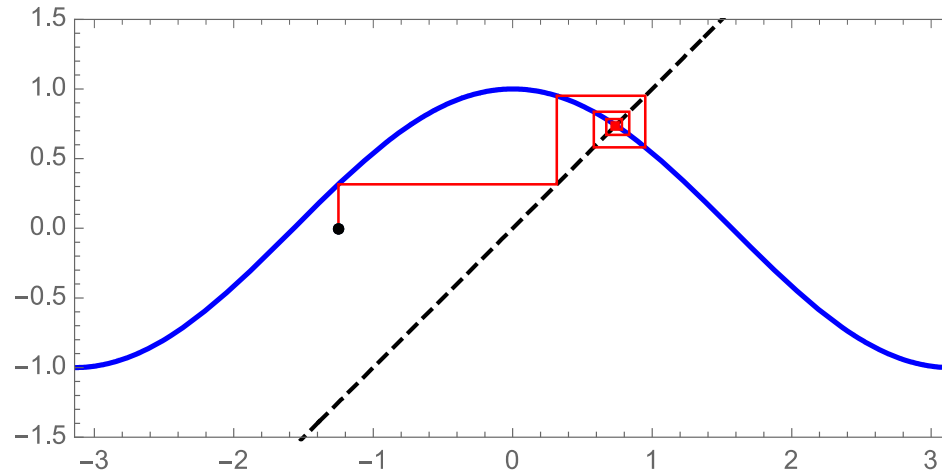
1. Go to position x_k on the horizontal axis
2. Draw a vertical line up to $y = F(x_k)$
3. Draw a horizontal to $y = x$ and set $x_{k+1} = F(x_k)$
4. Repeat to get x_{k+2} , etc.



Example: cosine map

Consider $x_{k+1} = \cos x_k$

Cobweb diagrams for two different starting points:



The iteration converges to a
fixed point: $x^* = \cos x^*$
 ≈ 0.739

The logistic map

Logistic map, e.g. population dynamics: $x_{k+1} = rx_k(1 - x_k)$

- Two fixed points: $x^* = rx^*(1 - x^*) \implies rx^* \left[x^* - \left(1 - \frac{1}{r} \right) \right] = 0$
 $\implies x \in \{0, 1 - \frac{1}{r}\}$

- Jacobian linearization around equilibrium points:

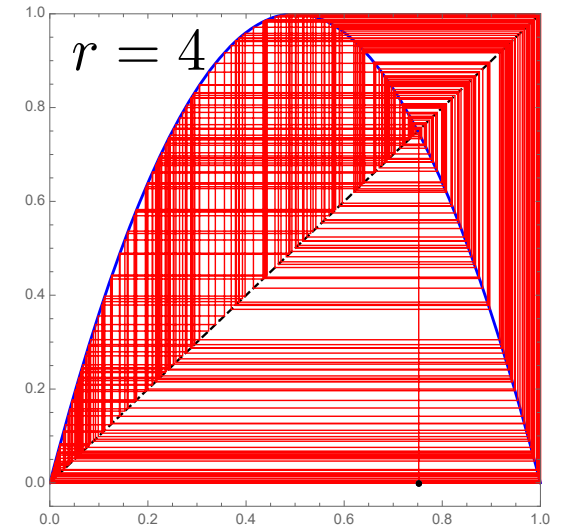
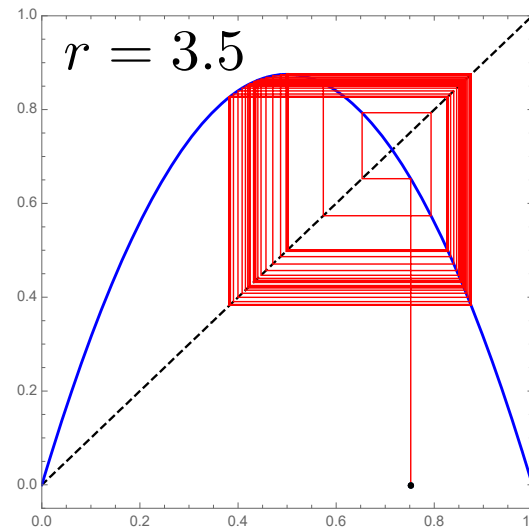
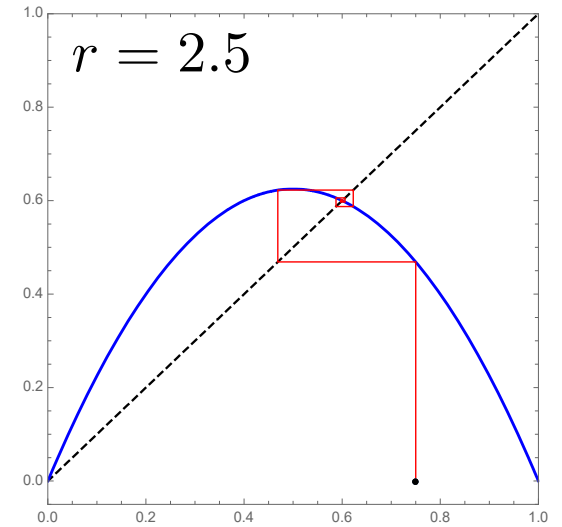
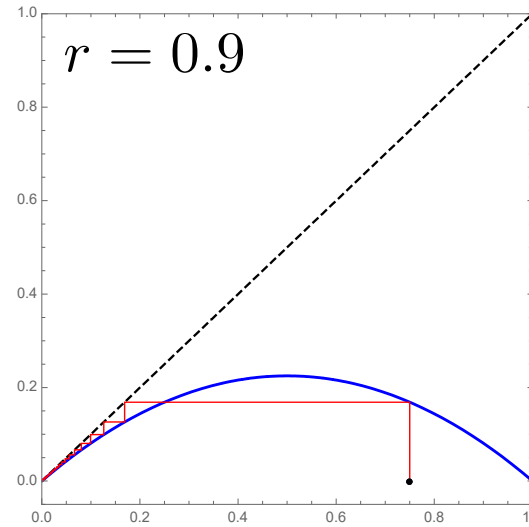
$$w_{k+1} = rw_k(1 - 2x^*) \implies \begin{cases} w_{k+1} = rw_k & \text{at } x^* = 0 \\ w_{k+1} = (2 - r)w_k & \text{at } x^* = 1 - \frac{1}{r} \end{cases}$$

- Behaviour near $x^* = 0$: population dies out if $r < 1$
population grows if $r > 1$
- What happens at the other equilibrium point?

Logistic map cobweb diagrams

$$x_{k+1} = rx_k(1 - x_k)$$

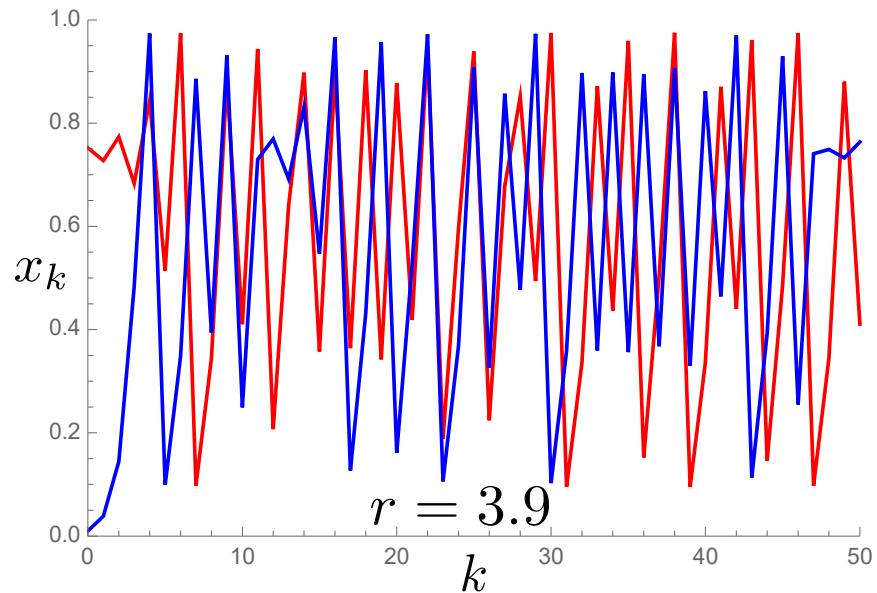
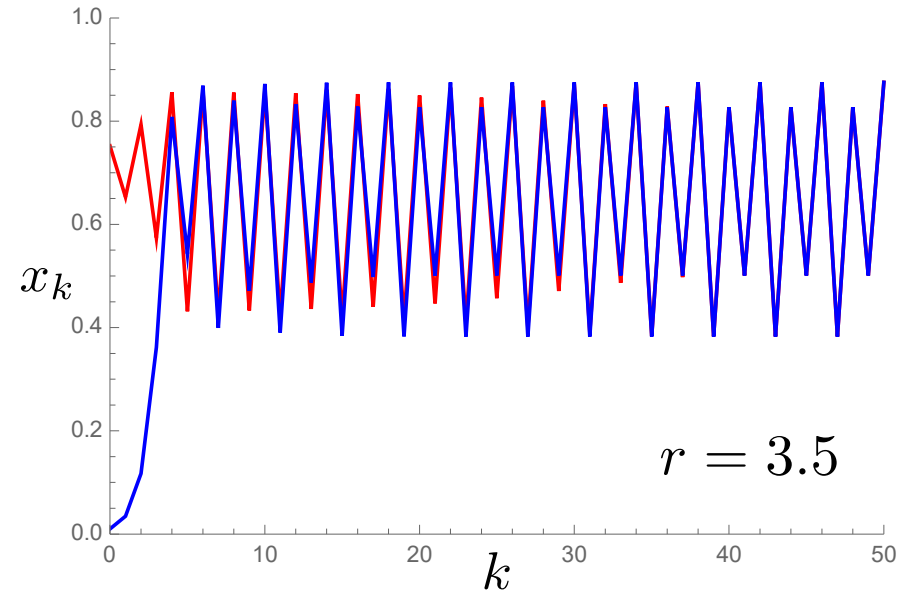
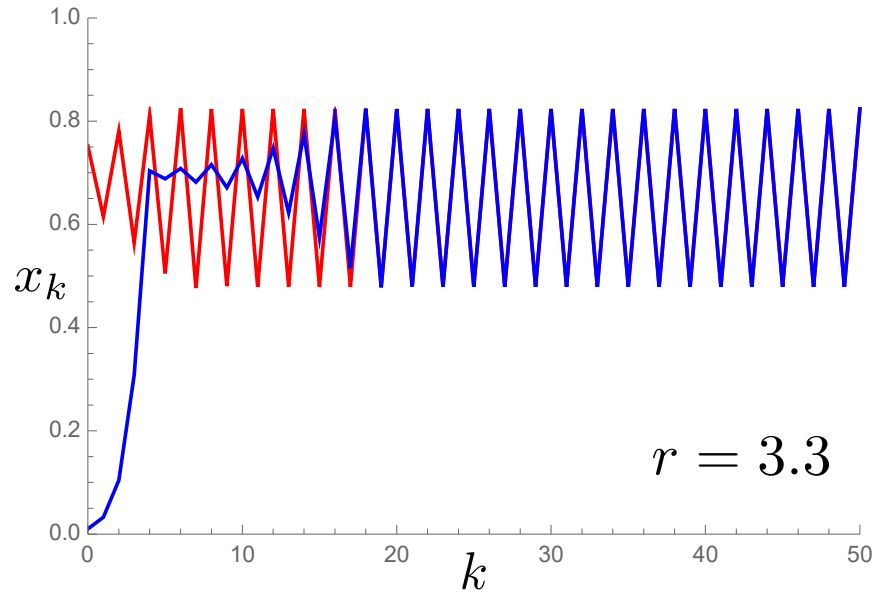
$$x_0 = 0.752$$



As r increases past 1, fixed point moves away from zero

With further increase in r , solution becomes periodic, then aperiodic

Logistic map simulations



- $r = 3.3$: trajectory is periodic
- $r = 3.5$: double-periodic
- $r = 3.9$: aperiodic
- Initial condition x_0 doesn't affect results

Behaviour of the logistic map

Consider points near $x^* = 1 - 1/r$ where the map's linearization is

$$w_{k+1} = (2 - r)w_k$$

- This equilibrium is stable if $|2 - r| \leq 1$, i.e. $1 \leq r \leq 3$
- For $r > 3$, $(2 - r) < -1$, so oscillations appear, with period doubling at fixed values of r as r increases
- The rate of doubling increases until r approaches ~ 3.57 , at which point the response ceases to be periodic
- As r continues to increase, the aperiodic behaviour continues with brief intermittent 'islands of stability' that are periodic

Periodicity of the logistic map

- Periodicity can be investigated by considering expressions like

$$x_{k+2} = F(F(x_k)) = r^2 x_k (1 - x_k) [1 - r x_k (1 - x_k)]$$

- Fixed points of this map are called 2-cycles (they have period 2) and solve the equation

$$x^* = F(F(x^*)) = r^2 x^* (1 - x^*) [1 - r x^* (1 - x^*)]$$
$$\implies x^* \in \left\{ 0, \quad 1 - \frac{1}{r}, \quad \frac{1}{2} \left(1 + \frac{1}{r}\right) \left[1 - \left(\frac{r-3}{1+r}\right)^{1/2}\right], \quad \frac{1}{2} \left(1 + \frac{1}{r}\right) \left[1 + \left(\frac{r-3}{1+r}\right)^{1/2}\right] \right\}$$

- A bifurcation occurs at $r = 3$, when the last 2 roots become real; linearizing around these roots \implies stable for $3 < r \leq 1 + \sqrt{6} \approx 3.45$
- Similar analysis is possible for 4-cycles, 8-cycles etc but becomes intractable

Orbit diagrams

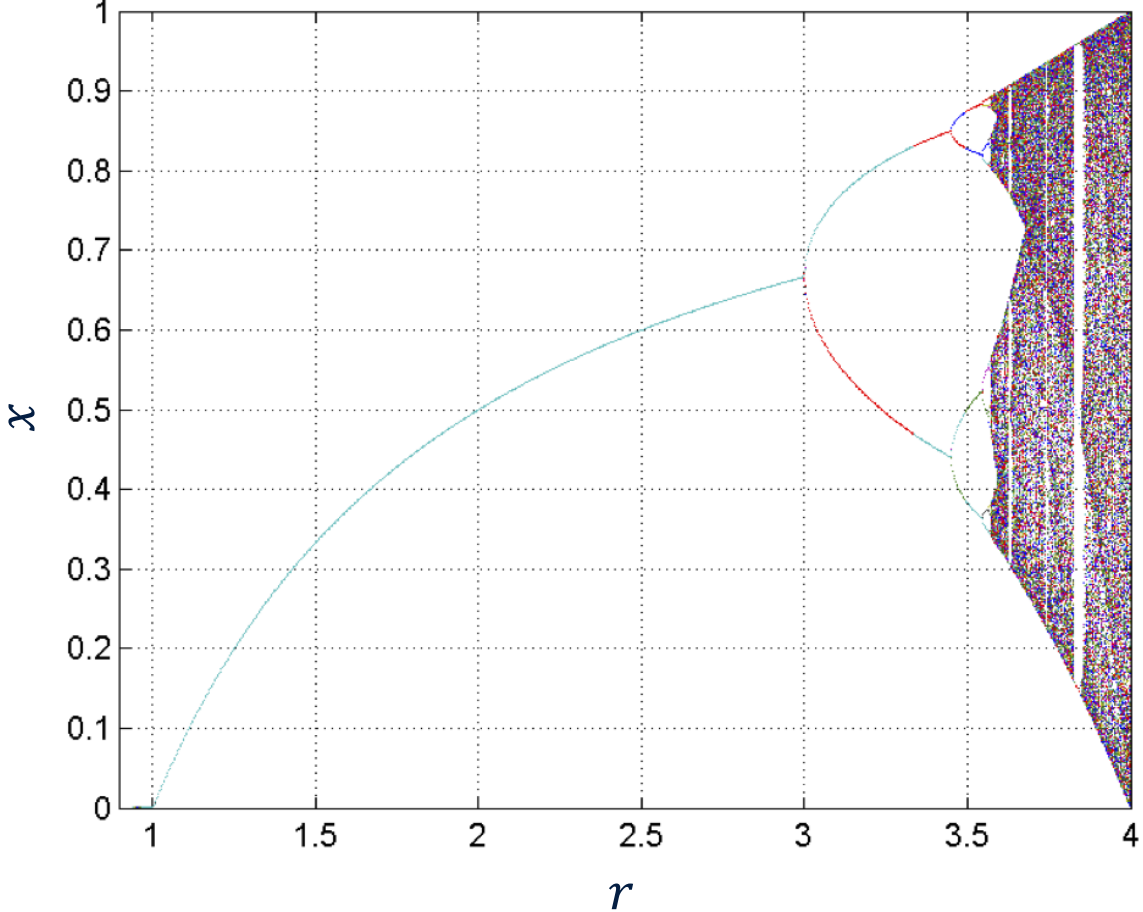
An orbit diagram displays how a map changes with respect to a parameter (and an initial value)

The process

1. Choose a value of r and a starting point x_0
2. Iterate the map m times (e.g. $m =$ a few hundred)
3. Record the values obtained when iterating another n times (e.g. $n =$ a few hundred), and plot them on the diagram
4. Change r and repeat

Several plots are given in the lecture notes

Orbit diagram for the logistic map



Notice the bifurcations and doubling, as well as the descent into aperiodicity

Chaos

The logistic map illustrates the phenomenon of dynamical **chaos**

Chaos is defined as **aperiodic** long-term behaviour in a **deterministic** system that has a **sensitive dependence on initial conditions**

- Aperiodic long-term behaviour: The trajectories never settle down to fixed points or periodic orbits
- Deterministic system: The trajectory is the solution of an equation with no noise - everything is certain and precise
- Sensitive to initial conditions: Trajectories that pass through points that are close together in phase space diverge with time

Lyapunov exponent

- The Lyapunov exponent measures how fast solutions that are initially close together in phase space diverge (i.e. it quantifies sensitivity to initial conditions)
- Consider the effect on long-term behaviour when perturbing the initial condition from x_0 to $x_0 + w_0$:

$$x_1 + w_1 = F(x_0 + w_0)$$

$$x_2 + w_2 = F(F(x_0 + w_0))$$

$$\vdots$$

$$x_k + w_k = F(\cdots F(x_0 + w_0))$$

$$w_1 = F(x_0 + w_0) - F(x_0)$$

$$w_2 = F(F(x_0 + w_0)) - F(F(x_0))$$

$$\implies$$
$$\vdots$$

$$w_k = F(\cdots F(x_0 + w_0)) - F(\cdots F(x_0))$$

- The Lyapunov exponent λ scales the growth rate of w_k : $|w_k| \approx |w_0|e^{\lambda k}$

Determining Lyapunov exponents

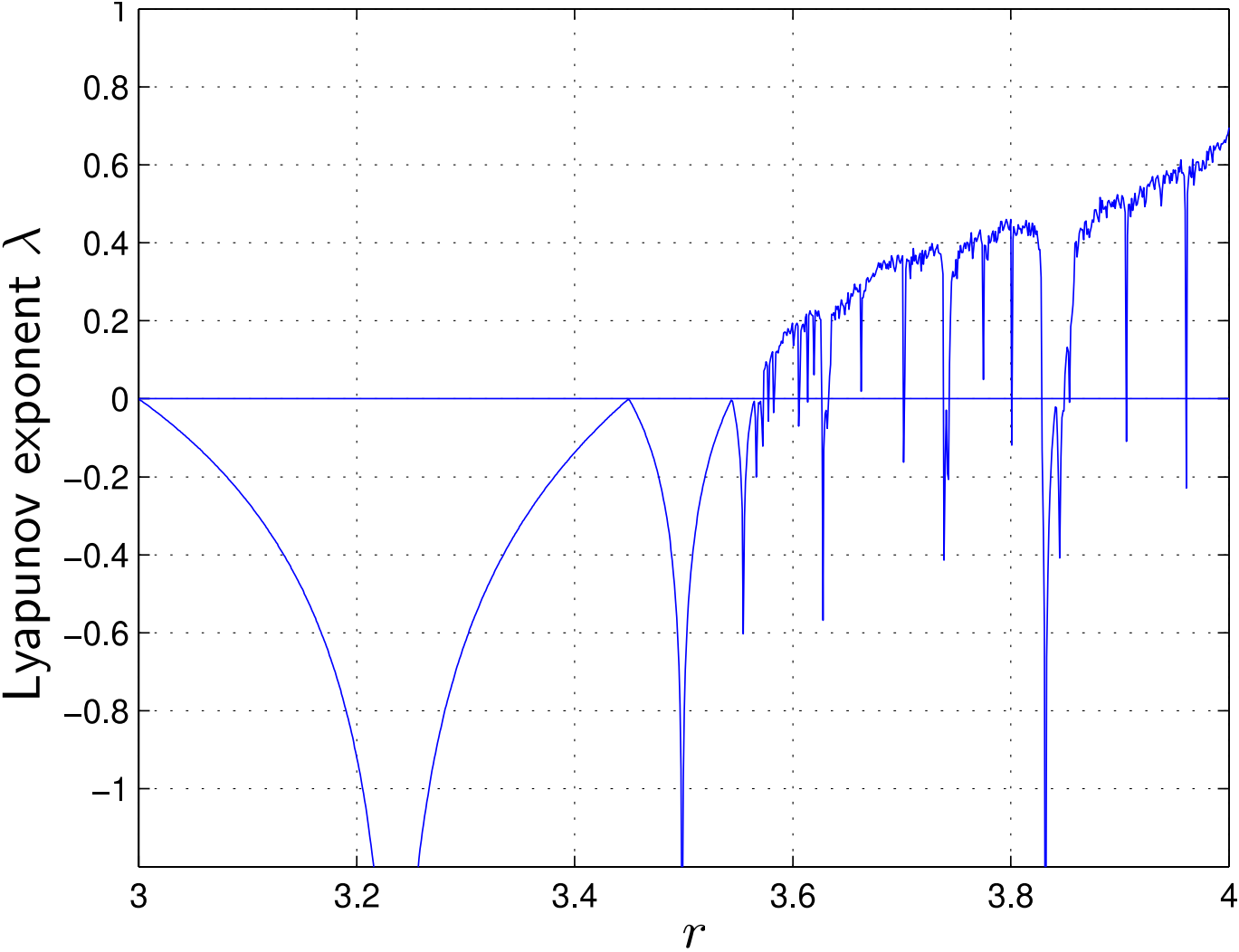
Compute Lyapunov exponents using linearisation:

$$\begin{aligned}w_1 &= F(x_0 + w_0) - x_1 \\ &\approx F(x_0) + DF(x_0)w_0 - x_1 = DF(x_0)w_0 \\ w_2 &\approx DF(x_1)w_1 \\ &\approx DF(x_1)DF(x_0)w_0 \\ &\vdots \\ w_k &\approx \prod_{i=0}^{k-1} DF(x_i)w_0\end{aligned}$$

The Lyapunov exponent is then given by

$$\lambda = \lim_{k \rightarrow \infty} \lim_{|w_0| \rightarrow 0} \frac{1}{k} \ln \left| \frac{w_k}{w_0} \right| \approx \frac{1}{k} \sum_{i=0}^{k-1} \ln |DF(x_i)| \quad (\text{accurate for large } k)$$

Lyapunov exponents for the logistic map



Dips in the exponent indicate periodic solutions

Chaos in flows

Chaos can also occur in differential systems

The **Lorenz equations**, which model convection in the atmosphere, are a well-known system that exhibits chaos:

$$\begin{aligned}\dot{x} &= \sigma(y - x) & \sigma > 0 : \text{Prandtl number} \\ \dot{y} &= rx - y - xz & r > 0 : \text{Rayleigh number} \\ \dot{z} &= xy - bz & b > 0 : \text{constant parameter}\end{aligned}$$

- If $r < 1$, the only equilibrium is the origin
- If $r > 1$ two more equilibria appear via a pitchfork bifurcation:

$$(x^*, y^*, z^*) \in \left\{ \left(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1 \right), \left(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1 \right) \right\}$$

Volume contraction in the Lorenz equations

From the Lorenz equations

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz\end{aligned}$$

we can compute the divergence of the flow:

$$\nabla \cdot \mathbf{f} = \frac{\partial \sigma(y - x)}{\partial x} + \frac{\partial rx - y - xz}{\partial y} + \frac{\partial xy - bz}{\partial z} = -(1 + b + \sigma)$$

- Since $\nabla \cdot \mathbf{f} < 0$, we have, for any control volume V :

$$\int_V \nabla \cdot \mathbf{f} dV = \oint_{\partial V} \mathbf{f} \cdot d\mathbf{S} = \text{flow out} - \text{flow in} < 0$$

- So trajectories converge to a zero-volume region of phase space but this zero-volume solution is neither a point or a limit cycle!

Lorenz equations: stability 1

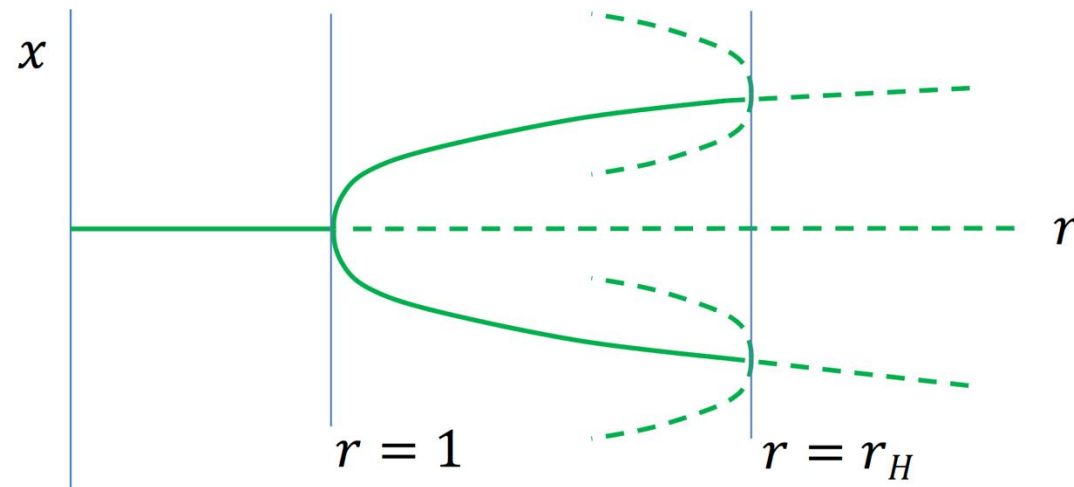
- A solution of the Lorenz equations cannot have unstable equilibrium points or unstable periodic orbits – both of these types of solution imply expansion of the state space, not contraction
- Any fixed points must therefore be stable or saddles
if there are limit cycles, they must be stable
- Linearization reveals that the origin is a stable node for $r < 1$
and a saddle for $r > 1$
- For $r < 1$ Lyapunov analysis shows that the system is globally asymptotically stable
– there are no limit cycles and all trajectories fall into the origin

Lorenz equations: stability 2

For $r > 1$ two new equilibria appear: assume $\sigma - b > 1$ and let

$$r_H = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}$$

- for $1 < r < r_H$ the two new equilibria are stable
- at $r = r_H$ they undergo a Hopf bifurcation
- for $r > r_H$ there is a saddle point (and no attractors nearby)



Strange attractor

Here we have a situation where:

- the volume occupied by neighbouring trajectories in phase space is always contracting
- there are no stable equilibrium points
- there are no stable limit cycles for $r > r_H$ (Lorenz proved this)
- trajectories cannot go to infinity

We conclude that there must be a zero-volume object that attracts the trajectories - we call this a **strange attractor**

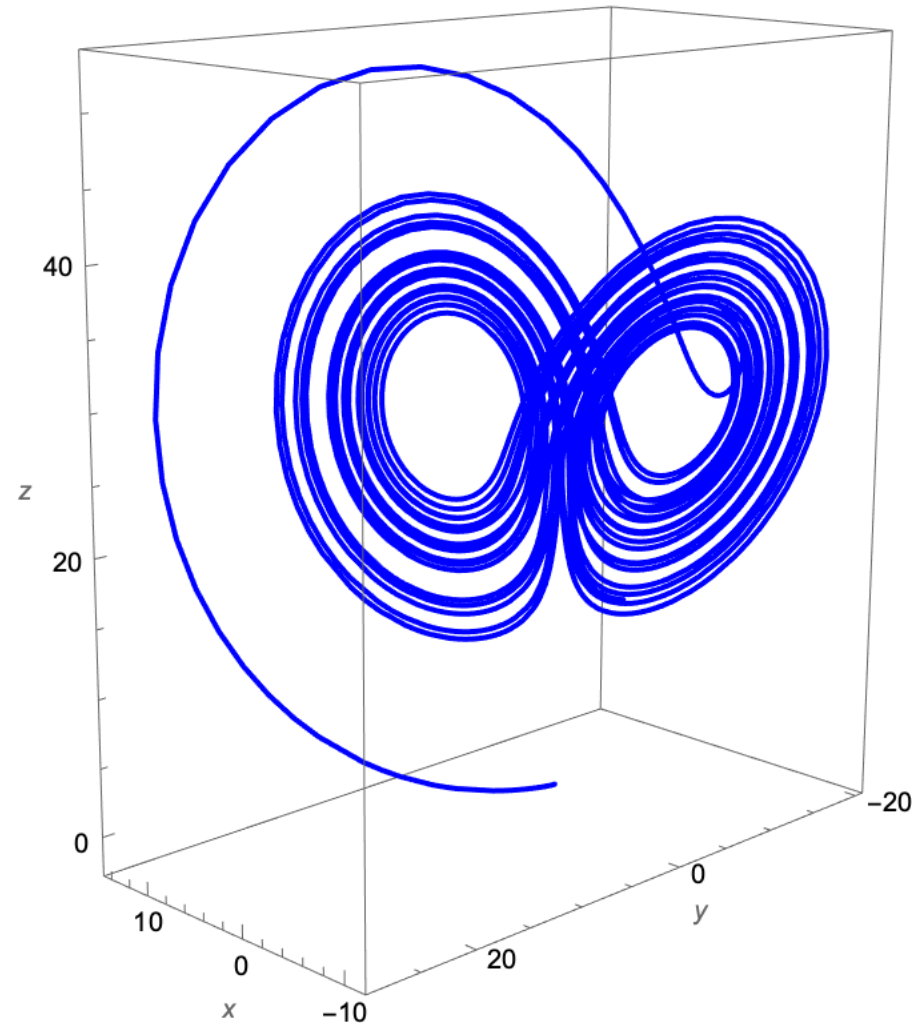
Lorenz equations: strange attractor

Plot of solution trajectory for $\sigma = 3$, $r = 29.4$, $b = 1$
initial condition $(x, y, z) = (0.1, 0.1, 0.1)$

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = rx - y - xz$$

$$\dot{z} = xy - bz$$



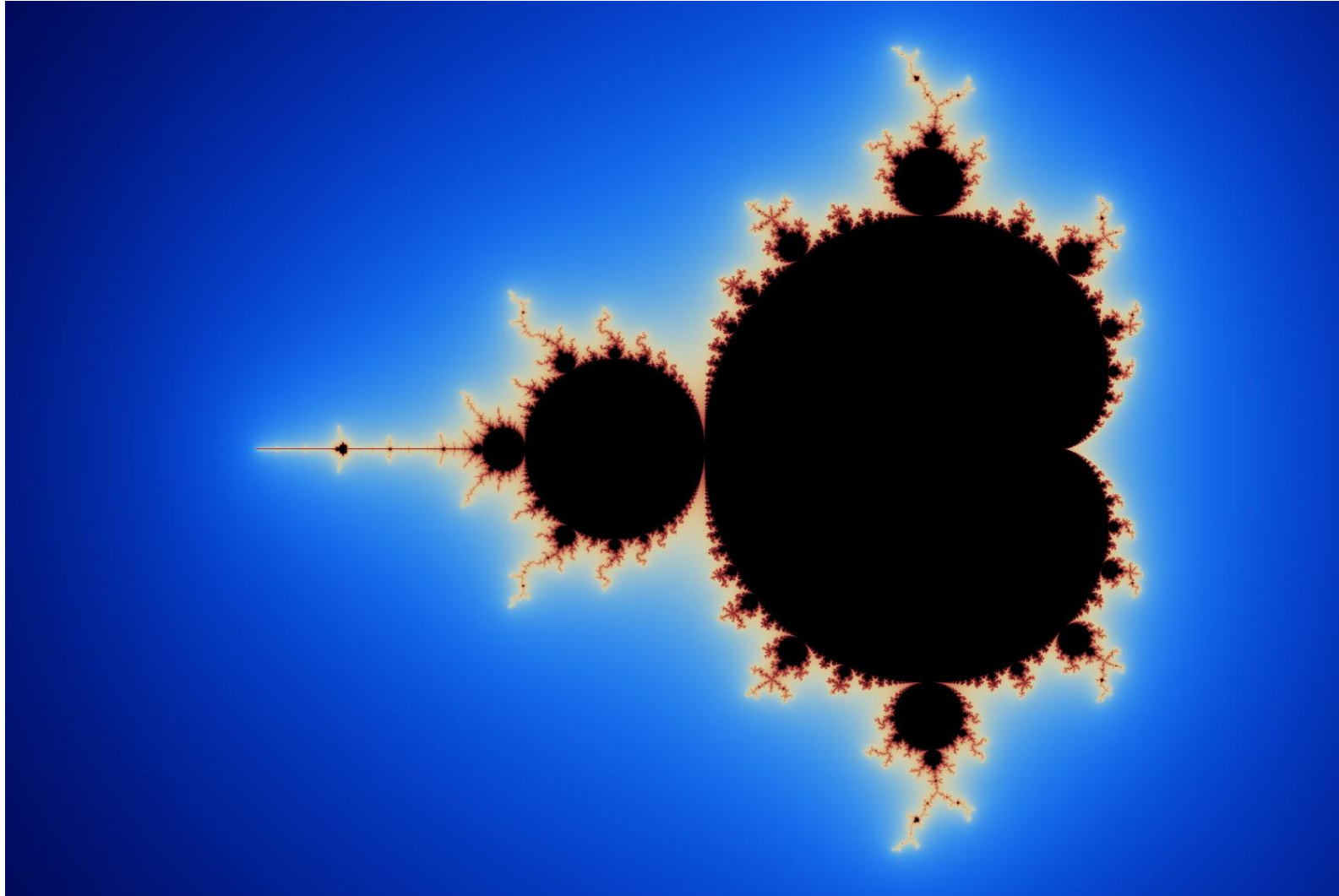
The Mandelbrot set

We now revisit a map defined over the complex numbers by

$$z_{k+1} = z_k^2 + c, \quad c \in \mathbb{C}$$

- The point c belongs to the **Mandelbrot set** if this iteration remains bounded for all k when starting at $z_0 = 0$
- To illustrate the set, we use different colours to show the rate of divergence at c
- The Mandelbrot set is a strange attractor for the map; colours indicate how close the points are to the attractor

The Mandelbrot set



Questions?

1. Introduction to Dynamical Systems. Examples from different fields. Maps, flows and questions of interest. A geometric way of thinking about ODEs.
2. Phase space, equilibria. Stability and linearized stability. Saddles, nodes, foci and centres.
3. The stable, unstable and centre subspaces. Hartman-Grobman theorem.
4. Lyapunov functions. Gradient and Hamiltonian Systems. Vector fields possessing an integral.
5. Invariance. La Salle's principle. Domains of attraction. Poincaré-Bendixson theorem
6. Limit sets, attractors, periodic orbits, limit cycles. Index theory. Poincaré maps.
7. Saddle-node, transcritical, pitchfork bifurcations. Hopf bifurcations.
8. Logistic map. Chaos. Lorenz equations.